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ON FREE OSCILLATIONS OF A VISCOUS INCOMPRESSIBLE FLUID IN SEMI-INFINITE CHANNEL*

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The motion of a viscous incompressible fluid in a semi-infinite channel is considered in the linearized theory of free interaction /1 - 3/. At some distance from the intake the independently developing on both side walls boundary layers begin to interact over the potential main body of the stream /4,5/. Concurrently in the channel appear symmetric and antisymmetric oscillations with a wave length of the order of the channel width. For this and other perturbations the first modes can be either stable or unstable, but the region of instability for symmetric oscillations is shifted to the side of lower frequencies. In a shift into the channel, the symmetric perturbations become longer-wave by comparison with antisymmetric ones. When the ratio of distance from the inlet to the channel width becomes of the order of the Reynolds number, a degeneration of the symmetric perturbations occurs, which is predicted by the theory of free interaction.

1. Asymptotic equations. Let t^* be the time, x^* and y^* be the Cartesian coordinates, u^* and v^* be the components of velocity vector, p^* be the pressure, and ρ^* the density (constant). Let us consider a channel, whose walls are specified by the equations $y^* = \mp {}^1/_2 b^*$. We assume for definiteness that at its inlet $x^* = 0$ a uniform stream $U_\infty^* = \mathrm{const}$ is realized. As shown in /6,7/, this obvious, at first glance, condition is artifical, since its fulfillment requires some mechanism, which would neutralize the small pressure gradients induced by the boundary layers. The result is the presence of weak vorticity in the inviscid part of the stream. Subsequently it will be sufficient to know only the principal term of the inviscid solution in the intake neighborhood, hence the consideration of selection of the most natural condition, at $x^* = 0$ can be omitted.

Assuming the Reynolds number higher than unity, we determine it by the channel width b^* velocity U_∞^* and kinematic viscosity ν^*. We shall consider that the free interaction of boundary layers with the uniform stream in both walls neighborhood at a distance of order $O(\mathrm{Re}\,\delta^2 b^*)$ from their boundaries can be described by the three-stage model /1 - 3/ of flow with the characteristic longitudinal dimension $O(\Delta b^*)$, which include the external regions 1 and 2 of the potential velocity field with equal scales in the longitudinal and transverse directions, the intermediate regions 3 and 4 of local inviscid but vortex boundary layer of thickness $O(\delta b^*)$, and the narrow nearest to the walls region 5 and 6 of width $O(\mathrm{Re}^{-1/3}\delta^{1/3}\Delta^{1/3}b^*)$, where the effect of viscosity is substantial. Since the length of the region of free interchange always exceeds the boundary layer thickness, hence $\delta \ll \Delta$. The parameters of fluid in each of the regions will be represented in the form of asymptotic expansions.

We denote by x_e the relative distance of order unity from the inlet, by p_∞ the constant part of the dimensionless pressure, and by U_B the function by means of which is established the profile of the longitudinal component of velocity in the Blasius solution. Then in regions 1 and 2 we have

$$
\begin{aligned}
&t^* U_\infty^*/b^* = \mathrm{Re}^{1/3}\delta^{2/3}\Delta^{2/3} t, \qquad x^*/b^* = \mathrm{Re}\,\delta^2 x_e + \Delta x \\
&y^*/b^* = \mp {}^1/_2 + \Delta y_{1,2} \\
&u^*/U_\infty^* = 1 + \varepsilon u_{1,2}(t, x, y_{1,2}) + \ldots \\
&v^*/U_\infty^* = \varepsilon v_{1,2}(t, x, y_{1,2}) + \ldots, \quad p^*/(\rho^* U_\infty^{*2}) = p_\infty + \varepsilon p_{1,2}(t, x, y_{1,2}) + \ldots
\end{aligned} \tag{1.1}
$$

In regions 3 and 4 the relations are satisfied

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$$t^*U_\infty^*/b^* = \mathrm{Re}^{1/3}\delta^{2/3}\Delta^{2/3}t, \quad x^*/b^* = \mathrm{Re}\,\delta^2 x_e + \Delta x \tag{1.2}$$
$$y^*/b^* = \mp {}^1/_2 + \delta y_{3,4}$$
$$u^*/U_\infty^* = U_B(y_{3,4}) + \mathrm{Re}^{-1/3}\delta^{-2/3}\Delta^{1/3}u_{3,4}(t, x, y_{3,4}) + \ldots$$
$$v^*/U_\infty^* = \mathrm{Re}^{-1/3}\delta^{1/3}\Delta^{-2/3}v_{3,4}(t, x, y_{3,4}) + \ldots$$
$$p^*/(\rho^* U_\infty^{*2}) = p_\infty + \mathrm{Re}^{-2/3}\delta^{-4/3}\Delta^{2/3}p_{3,4}(t, x, y_{3,4}) + \ldots$$

and in regions 5 and 6 the following formulas are valid

$$t^*U_\infty^*/b^* = \mathrm{Re}^{1/3}\delta^{2/3}\Delta^{2/3}t, \quad x^*/b^* = \mathrm{Re}\,\delta^2 x_e + \Delta x \tag{1.3}$$
$$y^*/b^* = \mp \frac{1}{2} + \mathrm{Re}^{-1/3}\delta^{1/3}\Delta^{1/3}y_{5,6}$$
$$u^*/U_\infty^* = \mathrm{Re}^{-1/3}\delta^{-2/3}\Delta^{1/3}u_{5,6}(t, x, y_{5,6}) + \ldots$$
$$v^*/U_\infty^* = \mathrm{Re}^{-2/3}\delta^{-1/3}\Delta^{-1/3}v_{5,6}(t, x, y_{5,6}) + \ldots$$
$$p^*/(\rho^* U_\infty^{*2}) = p_\infty + \mathrm{Re}^{-2/3}\delta^{-4/3}\Delta^{2/3}p_{5,6}(t, x, y_{5,6}) + \ldots$$

The scale coefficients δ and Δ, and also the amplitude factor ε are expressed by some powers of the Reynolds numbers, which are changed at the change at modes of flow. Below we assume that

$$\mathrm{Re}^{-1/3}\delta^{-2/3}\Delta^{1/3} \ll 1, \quad \varepsilon \ll 1$$

Parameter $\delta \ll 1$ is in all cases, except the limit one which is realized when the boundary layers join at the axis of the channel. For the limit mode $\delta = O(1)$.

Let us substitute the expansions (1.1)—(1.3) into the Navier—Stokes equations for the incompressible fluid and, wherever possible, carry out partial integration of relations that obtain for the sought functions of the first approximation. As the result, for the 1 and 2 regions we find

$$u_{1,2} = -p_{1,2}, \quad \frac{\partial U_{1,2}}{\partial x} + \frac{\partial v_{1,2}}{\partial y_{1,2}} = 0, \quad \frac{\partial v_{1,2}}{\partial x} + \frac{\partial p_{1,2}}{\partial y_{1,2}} = 0 \tag{1.4}$$

In regions 3 and 4 the solution is

$$u_{3,4} = A_{3,4}(t, x)\frac{dU_B}{dy_{3,4}}, \quad v_{3,4} = -\frac{\partial A_{3,4}}{\partial x}U_B(y_{3,4}) \tag{1.5}$$
$$p_{3,4} = \begin{cases} p_{3,4}(t, x), & \mathrm{Re}^{-1/3}\delta^{-8/3}\Delta^{7/3} \gg 1 \\ P_{3,4}(t, x) + \dfrac{\partial^2 A_{3,4}}{\partial x^2}\displaystyle\int_0^{y_{3,4}} U_B^2(Y)\,dY, & \mathrm{Re}^{-1/3}\delta^{-8/3}\Delta^{7/3} = 1, \end{cases}$$

with arbitrary functions $p_{3,4}(t, x)$, $P_{3,4}(t, x)$ and $A_{3,4}(t, x)$. The lower expressions for the surplus pressure, which contains the dependence on the second derivative $\partial^2 A_{3,4}/\partial x^2$, that is proportional to the curvature of the streamline, is used only in the case of limit mode when $\delta = O(1)$.

Finally, for regions 5 and 6 we have the following Prandtl equations

$$\frac{\partial u_{5,6}}{\partial x} + \frac{\partial v_{5,6}}{\partial y_{5,6}} = 0, \quad \frac{\partial p_{5,6}}{\partial y_{5,6}} = 0 \tag{1.6}$$
$$\frac{\partial u_{5,6}}{\partial t} + u_{5,6}\frac{\partial u_{5,6}}{\partial x} + v_{5,6}\frac{\partial u_{5,6}}{\partial y_{5,6}} = -\frac{\partial p_{5,6}}{\partial x} + \frac{\partial^2 u_{5,6}}{\partial y_{5,6}^2}$$

The boundary conditions at the walls are obvious: $u_{5,6} = v_{5,6} = 0$ for $y_{5,6} = 0$. The lacking boundary conditions are determined, as usual, by the requirement of merging solutions for neighboring regions. Leaving to the end of the article the study of limit properties of the stream with $\delta = O(1)$ we find that at the boundary of 1—3 and 2—4 the following formulas hold:

$$\varepsilon v_{1,2}(t, x, \mp {}^1/_2) = -\mathrm{Re}^{-1/3}\delta^{1/3}\Delta^{-2/3}\frac{\partial A_{3,4}}{\partial x} \tag{1.7}$$
$$\varepsilon p_{1,2}(t, x, \mp {}^1/_2) = \mathrm{Re}^{-2/3}\delta^{-4/3}\Delta^{2/3}p_{3,4}(t, x)$$

Selecting the constant $\lambda = 0.3321$ on the asymptotic representation of the Blasius solution near the wall, at the boundaries of 3—5 and 4—6 we obtain

$$p_{5,6} = p_{3,4}(t, x) \tag{1.8}$$
$$u_{5,6} \mp \lambda x_e^{-1/2}y_{5,6} \to \pm \lambda x_e^{-1/2}A_{3,4}(t, x), \quad y_{5,6} \to \pm\infty$$

From (1.7) follow the requirements

$$\varepsilon = \mathrm{Re}^{-1/5}\delta^{1/2}\Delta^{-3/2}, \quad \mathrm{Re}^{1/5}\delta^{5/2}\Delta^{-1/2} = 1 \tag{1.9}$$

Thus one of the parameters δ or Δ remains arbitrary, and it is by this that the present analysis differs from the one used in the theory of free interaction /1 - 3/. This is explained by the lack of tie between the thickness of unperturbed boundary layers and the introduction of the Reynolds number over the channel width. If we set now $\delta = \mathrm{Re}^{-\alpha}$, then

$$\Delta = \mathrm{Re}^{(1-5\alpha)/4}, \quad \varepsilon = \mathrm{Re}^{(\alpha-1)/2} \tag{1.10}$$

As long as $1 > \alpha > {}^1/_5$, the external regions 1 and 2 width is considerably smaller than the channel width, and the problems at both walls are formulated separately, and for all perturbations the condition of their damping in the transverse direction is introduced as $y_1 \to \infty$ and $y_2 \to -\infty$, respectively. This case was in fact considered, when the variables $y_{1,2}$ were introduced in the third of formulas (1.1). When α attains the value 1/5 (which gives the distance $\mathrm{Re}^{3/5} x_e b^*$ from the inlet), the equalities $\Delta = 1, \varepsilon = \mathrm{Re}^{-2/5}$ follow from (1.10). The first of these means that the dimensions of regions 1 and 2 become of the order of channel width, and they overlap, forming a single potential core for both boundary layers, through which their interaction is effected. When $\delta = \mathrm{Re}^{-1/5}$ and $\Delta = 1$ it is expedient to joint the regions 1 and 2 in region 0 with the transverse coordinate y defined by formula $y = \mp{}^1/_2 + y_{1,2}$. Such shift does not alter the form of Eq.(1.4) and of boundary conditions (1.7). From here, the previous conditions of damping of all perturbations in the transverse for the independents regions 1 and 2 are replaced by the boundary value problem for Eq.(1.4) which is satisfied by functions u_0, v_0 and p_0 in the band of region 0.

The flow mode which is described by the mathematical model with five-stage structure of velocity field was considered in /4,5/. The analysis of that mode reduces to the simultaneous integration of Eqs.(1.4) and two systems of Prandtl equations (1.6). The relation of their solutions is achieved using common functions $p_{5,6}(t, x) = p_{3,4}(t, x)$ and $A_{3,4}(t, x)$ which appear in the boundary conditions (1.7) and the limit conditions (1.8).

2. Free oscillations. In conformity with (1.4) functions v_0 and p_0 in region 0 satisfy the Laplace equation and are harmonically conjugate. We denote by a the amplitude of perturbations, and by b and d the arbitrary constants, and define the solution as

$$(v_0, p_0) = a\,[be^{ky} + de^{-ky},\ i\,(-be^{-ky} + de^{ky})]\, e^{i(\omega t + kx)} \tag{2.1}$$

In region 5 and 6 we represent the sought parameters of fluid in the form

$$(u_{5,6} \mp \mu_e y_{5,6},\ v_{5,6},\ p_{5,6},\ A_{3,4}) = a\left[-\frac{df_{5,6}}{dy_{5,6}},\ ikf_{5,6},\ \beta_{5,6},\ \gamma_{5,6}\right] e^{i(\omega t + kx)} \tag{2.2}$$

which for $a \ll 1$ gives the corresponding small deviations for the Blasius solution for the boundary layers at the lower and upper walls of the vessel if $\mu_e = \lambda x_e^{-1/2}$. The values of constants $\beta_{5,6}, \gamma_{5,6}$ will be established below.

The linearization of Prandtl equations (1.6) and the derivation of ordinary differential equations for functions f_5 and f_6 and their integration follow the basis outlined in /8/. We present the final result

$$f_{5,6}(z_{5,6}) = \mp \mu_e^{-1/3} (ik)^{2/3} \beta_{5,6} \left[\frac{d\,\mathrm{Ai}(\zeta)}{dz_{5,6}}\right]^{-1} \int\limits_{\zeta}^{z_{5,6}} dz \int\limits_{\zeta}^{z} \mathrm{Ai}(z')\, dz' \tag{2.3}$$

where the complex variables $z_{5,6} = \zeta \pm (ik\mu_e)^{1/3} y_{5,6}$ with $\zeta = i^{1/3}\omega\,(\mu_e k)^{-2/3}$, and $\mathrm{Ai}(z)$ is the Airy function. These expressions automatically satisfy the conditions of fluid sticking $u_{5,6} = v_{5,6} = 0$ when $y_{5,6} = 0$ $(z_{5,6} = \zeta)$. For the selection of regular branch of function $k^{1/3}$ appearing in the definition of $z_{5,6}$ we make in plane k a slit along the positive imaginary semiaxis and set $-3\pi/2 \leqslant \arg k = \vartheta_k \leqslant \pi/2$.

One can verify that solution (2.1) satisfies the boundary conditions (1.7) only then, when the constants $\beta_{5,6}, \gamma_{5,6}$ are linked by the following relations:

$$\begin{aligned} &(\beta_5 + \beta_6)\,\mathrm{sh}\,{}^1/_2 k + k\,(\gamma_6 - \gamma_5)\,\mathrm{ch}\,{}^1/_2 k = 0 \\ &(\beta_6 - \beta_5)\,\mathrm{ch}\,{}^1/_2 k + k\,(\gamma_5 + \gamma_6)\,\mathrm{sh}\,{}^1/_2 k = 0 \end{aligned} \tag{2.4}$$

Further two relations

$$\begin{aligned} &\mu_e^{4/3}\Phi(\zeta)\,\gamma_{5,6} = \pm (ik)^{1/3}\beta_{5,6} \\ &\Phi(\zeta) = \frac{d\,\mathrm{Ai}(\zeta)}{dz}\left[\int\limits_{\zeta}^{\infty} \mathrm{Ai}(z)\, dz\right]^{-1} \end{aligned} \tag{2.5}$$

between β_5 and γ_5, and β_6 and γ_6 are determined by the limit conditions $df_{5,6}(\mp\infty)\,/\,dy_{5,6} = \mp\mu_e\gamma_{5,6}$ for the unknown functions which follow from (1.8).

Equations (2.4) and (2.5) form a homogeneous system of four unknowns. However they admit decomposition with respect to the quantities $\beta_6+\beta_5$, $\gamma_6-\gamma_5$ and $\beta_6-\beta_5$, $\gamma_6+\gamma_5$ into two homogeneous second order systems, whose nontrivial solvability is defined by the equality to zero of the corresponding determinant, the existence of nontrivial solution of one system unavoidably implies its absence in the other system.

Thus at the inlet of the channel are possible two types of perturbations /5/: symmetric and antisymmetric with respective dispersion relations

$$\mu_e^{5/3}\Phi(\zeta) = k(ik)^{1/3}\operatorname{cth}{}^1\!/_2 k \quad (\beta_6=\beta_5,\ \gamma_6=-\gamma_5) \tag{2.6}$$
$$\mu_e^{5/3}\Phi(\zeta) = k(ik)^{1/3}\operatorname{th}{}^1\!/_2 k \quad (\beta_6=-\beta_5,\ \gamma_6=\gamma_5)$$

Each type of perturbations possesses his own frequency-wave spectrum, the two spectra provide a full set of natural frequencies and wavenumbers of free oscillations. In any case one of the constants in solution (2.2) remains arbitrary.

3. Properties of spectra. Using the results related to the free interaction of a boundary layer on the insulated plate /9/, we can state that to each specified k (or ω) in the complex plane ζ corresponds a countless multiplicity of roots lying in the neighborhood of the negative real semiaxis. We denote $\arg\zeta = \vartheta = \pi + \vartheta'$ and set $|\zeta|\to\infty$, as $\vartheta'\,|\zeta|^{3/2}\to 0$. On the basis of the asymptotic representation of the Airy function that remain continuous at points of the real negative semiaxis, from (2.6) we obtain

$$|\zeta|^{1/4}\left[\cos\left(\frac{2}{3}|\zeta|^{3/2}+\frac{\pi}{4}\right) - i\vartheta'|\zeta|^{3/2}\sin\left(\frac{2}{3}|\zeta|^{3/2}+\frac{\pi}{4}\right)\right] = \tag{3.1}$$
$$-\sqrt{\pi}\,\mu_e^{-5/3}k(ik)^{1/3}\times\begin{cases}\operatorname{cth}{}^1\!/_2 k, & \beta_6=\beta_5\\ \operatorname{th}{}^1\!/_2 k, & \beta_6=-\beta_5\end{cases}$$

from which directly follows the formulated statement. As number $j\to\infty$ and $|k|\to 0$, we have

$$|\zeta_j| = \left[\frac{3\pi}{2}\left(j+\frac{1}{4}\right)\right]^{2/3} \tag{3.2}$$
$$\vartheta_j = (-1)^{j+1}\frac{\sqrt{\pi}}{2}\mu_e^{-5/3}\left[\frac{3\pi}{2}\left(j+\frac{1}{4}\right)\right]^{-7/6}\times$$
$$\begin{cases}|k|^{1/3}\sin\frac{1}{3}\left(\vartheta_k+\frac{\pi}{2}\right), & \beta_6=\beta_5\\ |k|^{7/3}\sin\frac{7}{3}\left(\vartheta_k+\frac{\pi}{2}\right), & \beta_6=-\beta_5\end{cases}$$

The equality (3.2) means that for fixed real ω in the complex plane k the sequencies ζ_j for each of the dispersion relations corresponds an infinite succession of roots k_j exists in the neighborhood of ray $\arg k = -5\pi/4$ with the point of thickening at the coordinate origin. The presence of hyperbolic functions in the right-hand sides of (2.6) and (2.7) entail in the complex plane the existence k for symmetric and antisymmetric oscillations further three infinite sequencies of roots. The two of them lie along the slit sides $\arg k = \pi/2$ and $\arg k = -3\pi/2$, and one in the neighborhood of the imaginary semiaxis. In fact, $\operatorname{cth}{}^1\!/_2 k = 0$ when $k = \pm\pi(2n+1)$, $n = 0, 1, \ldots$; similarly $\operatorname{th}{}^1\!/_2 k = 0$, if $k = \pm 2\pi n i$, $n = 0, 1, \ldots$. Setting n reasonably large, we write

$$k_n = k_n' + e^{i\pi(l-1/2)}\times\begin{cases}(2n+1)\pi, & \beta_6=\beta_5\\ 2n\pi, & \beta_6=-\beta_5\end{cases},\quad l=-1,0,1 \tag{3.3}$$

where $\lim|k_n'| = 0$. We substitute in the left-hand sides of (2.6) function $\Phi(\zeta)$ by its limit value $\Phi(0)$, after which we represent both dispersion relations in the form

$$\mu_e^{5/3}\Phi(0) = {}^1\!/_2\, e^{i\pi/6}k_n^{4/3}k_n'$$

Since $\Phi(0) = -3^{2/3}\Gamma^{-1}(1/3)$, we finally obtain

$$k_n' = -2\cdot 3^{2/3}\pi^{-4/3}\mu_e^{5/3}\Gamma^{-1}(1/3)\,e^{i\pi(1/2-4/3l)}\times\begin{cases}(2n+1)^{-4/3}, & \beta_6=\beta_5\\ (2n)^{-4/3}, & \beta_6=-\beta_5\end{cases}$$

The sequence of roots k_n lying along the slit sides $\arg k = \pi/2$ and $\arg k = -3\pi/2$ generate two systems of waves propagating downstream from the source. Roots k_n from the sequence in the neighborhood of the negative imaginary semiaxis provide an infinite train of waves moving upstream.

Let now in accordance with the basic concepts of the linear theory of stability, the frequency ω assumes any complex values and at the same time, the wavenumber k is real. We shall calculate the critical value of k_* passing through which the imaginary part ω changes its sign. The condition $\mathrm{Im}\,\omega = 0$ defines running waves of Tolmin-Schlichting in which are accomplished neutral oscillations of fluid at constant time independent amplitude.

The trajectories of several first roots from the sequence (3.1) in plane ω when $\mu_e = 1$ are shown in Fig.1 for dispersion relations by solid and dash line respectively. The arrows indicate the direction in which k increases. The roots from the sequence (3.3) have no analogs in plane ω. The intersection by curves ω_1 of the axis of abscissas means that the amplitude of the first mode can degenerate with time, or exponentially increase. All remaining modes are stable. The region where $\mathrm{Im}\,\omega_1 > 0$ is substantially narrower for symmetric perturbations than for antisymmetric. As mentioned in /5/, the critical values of ω_* and k_* for Eqs.(2.6) can be found by simple recalculation of the respective critical values of the incompressible boundary layer on an isolated plate. Indeed, in both problems compared with each other, the arguments of the right-hand of dispersion relations are the same for real ω and k. The calculations for $\mu_e = 1$ yielded for symmetric perturbations $\omega_{*s} = -0.5736$, $k_{*s} = 0.1248$ and for antisymmetric perturbations $\omega_{*a} = -2.9270$, $k_{*a} = 1.4382$.

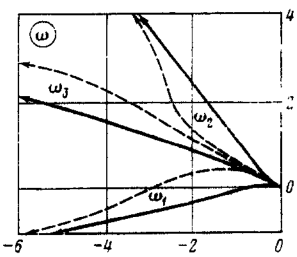

Fig.1

However, by definition parameter $\mu_e = \lambda x_e^{-1/2}$ defines the distance from the channel inlet. The nature of variation of curves ω_1 in dependence of μ_e is seen for symmetric oscillations in Fig.2 and of antisymmetric in Fig.3. With increasing μ_e, which corresponds to the shift towards the intake aperture and to independent boundary layers on the walls, the region of stability for both perturbation types expand and become of the same order. Conversely, the decrease of μ_e leads to a sharp narrowing of the low frequency region of stability symmetric perturbations in comparison with antisymmetric. It is, therefore, necessary to consider in greater detail the development of perturbations with the increase of distance from the inlet, when the boundary layer thickness becomes greater on the walls than $O(\mathrm{Re}^{-1/8}b^*)$.

4. The merging of boundary layers. When the distance from the inlet exceeds by order of magnitude $O(\mathrm{Re}^{3/8}b^*)$, it is necessary to set in formulas (1.10) the exponent $0 < \alpha < 1/5$, then $\Delta \gg 1$. As the solution in the potential region 0 is necessary to use not the full expressions in formulas (2.1), but only their principal terms proportional to integral powers of the small parameter Δ^{-1}. We have

$$v_0 = a\,[(b + d) + (b - d)\,\Delta^{-1}ky + {}^1/_2\,(b + d)\,\Delta^{-2}k^2y^2 + \ldots]\,e^{i(\omega t + kx)} \tag{4.1}$$
$$p_0 = a\,[(-b + d) + (b + d)\,\Delta^{-1}\,ky + {}^1/_2\,(-b + d)\,\Delta^{-2}k^2y^2]\,e^{i(\omega t + kx)}$$

where the previous variable y is related to the variable $y_0 = \mp{}^1/_2\Delta^{-1} + y_{1,2}$ by the equality $y = \Delta y_0$. Indeed, turning to the general case $\alpha \neq 1/5$ it is necessary to take into consideration that solution (2.1) of Eqs.(1.4) contains precisely the variable y_0. If one takes as valid relations (1.9) or makes equivalent supposition about the equality of scales of symmetric and

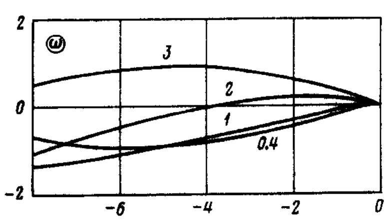

Fig.2

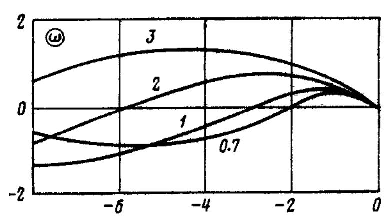

Fig.3

antisymmetric perturbation, then one can readily ascertain that the problem has only the trivial solution $b = d = 0$. Let us consider both solutions separately.

We begin with symmetric oscillations for which $b + d = 0$. Then from expansions (4.1) and conditions of merging (1.7) we obtain instead of (1.9) the new relations

$$\varepsilon = \mathrm{Re}^{-1/3}\delta^{1/3}\Delta^{4/3}, \quad \mathrm{Re}^{1/2}\delta^{3/2}\Delta^{-1/2} = 1$$

for the scale multipliers, whence let $\delta = \mathrm{Re}^{-\alpha}$ we find

$$\Delta = \mathrm{Re}^{1-5\alpha}, \quad \varepsilon = \mathrm{Re}^{-2\alpha} \tag{4.2}$$

For $\alpha = 1/5$ we revert to formulas (1.10), when it is possible to apply the developed above approach. If $0 < \alpha < 1/5$, then all reasoning is carried out similarly to the method set out in Sect.2. The single difference consist in the form of relation between constants $\beta_6 + \beta_5$ and $\gamma_6 - \gamma_5$. The second of Eqs.(2.4) is replaced by

$$(\beta_6 + \beta_5) + 2(\gamma_6 - \gamma_5) = 0 \tag{4.3}$$

As the result, the dispersion equation becomes

$$\mu_e^{5/3}\Phi(\zeta) = 2(ik)^{1/3} \tag{4.4}$$

The dependence (4.3) has a simple physical interpretation: in a dimentionless system of reference units the self-induced pressure

$$p_6 + p_5 = 2(A_3 - A_4)$$

is equal to double the shift of streamlines lying on both sides of the channel axis.

For antisymmetric oscillations $b - d = 0$. The expansions of (4.1) and the conditions of merging (1.7) instead of (1.9) provide

$$\varepsilon = \mathrm{Re}^{-2/3}\delta^{-4/3}\Delta^{5/3}, \quad \mathrm{Re}^{1/3}\delta^{5/3}\Delta^{-7/3} = 1$$

and, as the result of substitution of $\delta = \mathrm{Re}^{-\alpha}$ we further have

$$\Delta = \mathrm{Re}^{(1-5\alpha)/7}, \quad \varepsilon = \mathrm{Re}^{(\alpha-3)/7} \tag{4.5}$$

When $\alpha = 1/5$ equalities (4.5) are the same as (1.10). The lowering of α results in that the first of Eqs.(2.4) which binds the constants $\beta_6 - \beta_5$ and $\gamma_6 + \gamma_5$ changes as follows:

$$2(\beta_6 - \beta_5) + k^2(\gamma_6 + \gamma_5) = 0 \tag{4.6}$$

and leads to the dispersion equation of the form

$$\mu_e^{5/3}\Phi(\zeta) = \tfrac{1}{2}k^2(ik)^{1/3} \tag{4.7}$$

The physical meaning of the dependence (4.6) is evident: in conformity with it the self-induced pressure

$$p_6 - p_5 = \frac{1}{2}\frac{\partial^2(A_4 + A_3)}{\partial x^2}$$

is expressed in terms of curvature of streamlines lying on different sides of the axis of the channel. Correct within the coefficient of proportionality a similar representation of the surplus pressure is given by the lower line of (1.5).

The comparison of (4.2) and (4.5) shows that the wave length and the transverse dimension of the boundary layers 5 and 6 nearest to the wall which is proportional to the product $\mathrm{Re}^{-1/3}\delta^{1/3}\Delta^{1/3}$ for the symmetric perturbations arising in a plane channel increase with distance from its inlet much quicker that in the case of antisymmetric.

To complete the analysis expounded above, it remains to deal briefly with the limit case $\alpha = 0$, in which according with the asymptotic formulas (1.2), the boundary layers 3 and 4 merge, fill the whole channel and form a flow that is transitional to Poiseuille flow. For symmetric oscillations we have $\delta = 1$, $\Delta = \mathrm{Re}$, $\varepsilon = 1$. Since the boundary layers 5 and 6 increase the thickness to the channel width, by virtue of relation $\mathrm{Re}^{-1/3}\delta^{1/3}\Delta^{1/3} = 1$, the inviscid core disappears from the velocity field. Recalling formulas (1.5) we conclude from equality (4.3) that with the disappearance of the potential core 0 and of vortex regions 3 and 4, the symmetric part of pressure must be determined by the function which is discontinuous on the channel axis $y = 0$ and which specifies the shift of streamlines.

It is shown in /10,11/ that the theory, which includes the multistage structure of the continuous velocity field, of free interaction for the incompressible fluid is equivalent to the linear theory of stability which is based on the Orr-Sommerfeld equation, on condition that the critical layers are adjacent to streamlined walls. In the limit case, when the ratio of wave length and the distance from the inlet to the channel width becomes of the order of the Reynolds number, the critical layers 5 and 6 fill the whole channel; naturally the nature of such perturbations is not defined by the theory of free interaction. As the result, the dispersion relation (4.4) loses its validity.

Let us now turn to antisymmetric perturbations for which the parameters $\delta = 1$, $\Delta = \mathrm{Re}^{1/7}$, $\varepsilon = \mathrm{Re}^{-3/7}$ for $\alpha = 0$. The inviscid core of stream is maintained in connection with that $\mathrm{Re}^{-1/3}\delta^{-1/3}\Delta^{1/3} = \mathrm{Re}^{-2/7}$, the close to the wall layers 5 and 6 remain essentially thinner than the channel width. The distance from the inlet is estimated as before by $O(\mathrm{Re}\, b^*)$, the adduced scales arise in the linear problem of the stability of flow of the Poiseuille flow /11/.

Obviously they are characteristic also for the intermediate flow in which is accomplished the transition from the velocity distribution formed by two interlocked Blasius profiles, to a parabolic distribution. Since in the limit case the relation $Re^{-1/5}\delta^{-4/5}\Delta^{7/5} = 1$ is fulfilled, hence in formulas (1.5) it is necessary to use the lower expression for the self-induced pressure which varies across the inviscid kernel of the flow consisting of joined regions 3 and 4. The derivative of the surplus pressure along the coordinate y is determined by the velocity graph of the basic motion of fluid in the fixed cross section $x = \text{const}$. The arising in the formulated problem of dispersion relation differs from (4.7) only by the coefficient in the right-hand side.

In /12/ are presented data of numerical computation of the characteristic of flow stability at entry to the semiinfinite channel in relation to antisymmetric oscillations. In these computations the velocity profile of the basic stream was determined by integration of Prandtl equations throughout the region bounded by the solid walls. Since the discovered in /6,7/ vorticity was neglected in the cited paper, its results are rather of a qualitative character than quantitative, although the tendency of development of the profile development of the velocity longitudinal component of the unperturbed stream along the length of the channel was obtained as in the asymptotic theory. It is natural that the form of the velocity profile in the basic stream that is determined by the distance from the inlet, has a desicive effect on the characteristic of stability. It is important to note that the computations confirm the instability of the investigated fluid motion with respect to even more longwave perturbations for shift into the channel.

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